Optimal priority assignment with constraint

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ABSTRACT

Considered are optimal service policies for discrete-time priority queueing systems with two classes of customers. The arrival statistics are arbitrary, and the service requirements are geometrically distributed with class dependent rates. The optimization objective is to minimize the time-average line-length of one customer class, subject to a hard constraint on the time-average line-length of the other customer class. When there is a policy that satisfies the constraint it is shown that there is also an optimal non-idling randomized policy. The randomization depends neither on the past history nor present state of the system and is therefore easy to implement. Furthermore, the generating function for the two-dimensional line-length process is derived, which in turn enables the determination of the optimal bias factor.

RESUMÉ

Nous considérons des politiques de service optimales dans un système de files d'attente à temps discret avec deux classes de clients. Les processus des arrivées sont arbitraires et les temps de service requis sont géométriquement distribués, de taux dépendant de la classe. L'objectif est de minimiser le nombre moyen de clients d'une classe sous une contrainte portant sur le nombre moyen de clients de l'autre classe. On montre que parmi les politiques admissibles il existe une politique optimale aléatoire et conservative d'une structure particulièrement simple, où l'aléa est représenté par un tirage de Bernouilli (pile ou face) de paramètre constant q. La fonction génératrice de la distribution jointe du nombre de clients peut être obtenue comme solution d'une équation fonctionnelle, ce qui permet en retour de déterminer la valeur optimale de q. Ce calcul est effectué pour un processus des arrivées globalement géométrique.

I - INTRODUCTION

Studies of the dynamic control of queueing systems have become important in recent years because of the potential applications in performance analysis of computer and communication networks and systems. The general philosophy has been to search for a control policy depending on the past history and current state of the queueing system that minimizes a global cost [19], [2]. The global cost is usually expressed over the finite horizon, or over the infinite horizon with either a discounted or long-run average criterion.

In many cases, however, it is natural to impose a global constraint. For example, when controlling the arrival intensity to a queueing network, it may be of interest to minimize the average throughput subject to a constraint on the average time delay [5], [15].

Global constraints also arise naturally in the problem of optimally multiplexing voice and data packets onto a communication channel. For such systems, it is essential that the transmission time for the voice packets be kept small ; otherwise, the conversation that they represent would become incoherent. It is therefore of interest to find a control policy that maximizes the throughput of the data packets subject to a constraint reflecting the time delay of the voice packets.

In this paper we study optimal priority assignment for a queueing system when there is a global constraint to be met. The model is a natural one, and has potential use in the optimization of integrated voice/data systems or in other applications involving queueing of two user classes at a single facility.

Specifically, we analyze a priority assignment problem for a discrete-time queueing system with two classes of customers. Within time slots, customers arrive at an infinite capacity queue which is attended by a single server. The arrival patterns are i.i.d. from slot to slot. However, within a particular slot any arrival distribution is permitted, and arrivals from the two classes need not be independent.

At the beginning of each slot, the server's attention is given to a customer class on the basis of the control policy. Once a customer from class i (i = 0,1) has been selected to receive service, he either departs from the system during that slot with probability μ_i or remains in the system with probability $1 - \mu_i$. At most one customer departs from the system during a time slot. The service requirements are therefore distributed geometrically with class dependent rates μ_i, and are statistically independent from customer to customer. Note that a customer may be pre-empted as new priority decisions are made at the beginning of each slot.

This discrete-time queueing system has also been studied by Baras et al [2] ; the reader is referred there for an alternative discription. Our optimization criterion with the global constraint differs from theirs, and it leads to different analyses and results.

The control policy specifies which customer class will be selected for service at the beginning of each slot ; it is permitted to be randomized and to depend on the past history of the line-length process. Moreover, the priority discipline dictated by the control policy is not required to be work-conserving. A control policy is termed feasible if it satisfies the global constraint, namely, the long-run average line-length for type-0 customers must not exceed a specified threshold. A control policy is then optimal if it minimizes, within the class of feasible policies, the long-run average line-length of type-1 customers.

The primary result given in this paper is a proof of the existence of an optimal policy with a surprisingly simple structure. At the beginning of each slot, this optimal policy operates as follows : when only one customer class is present, a customer from that class is given the server's attention ; when both classes are present, a customer from a particular class is selected for service according to the outcome of a biased coin. The bias of the coin remains constant throughout the line-length process when an optimal randomized policy is in force. In particular, when a strict priority is employed (the bias of the coin either zero or one), we derive the steady-state moment generating function and obtain explicit expressions for the mean line-lengths. These results are of independent interest and do not seem to have been previously announced. Sidi and Segall [18] have investigated a similar discrete-time model under the assumption that the service rates for the customer classes are equal to 1 ($\mu_1 = \mu_0 = 1$).

When the general randomized policy (bias not necessarily zero or one) is employed, the steady-state generating function can be obtained by resolving a functional equation. Such functional equations frequently arise in queueing when, for example, two types of customers have to be distinguished. Recently, important advances have been made in obtaining a general methodology for solving such functional equations. For two-dimensional birth-death processes, Fayolle and Iasnogorodski [11], [12], showed that the resolution of the functional equation can be reduced to the solution of a Riemann-Hilbert type boundary value problem. More recently, Cohen and Boxma [7] have extended these results to (more general) two-dimensional random walks. These methodologies enable us to obtain an expression for the mean line-lengths in terms of the bias factor of the randomized policy in the case where the arrival process is globally geometric. This permits us to evaluate the optimal bias factor, i.e., to completely determine the optimal policy. Actually, following the method described in [7], a more general case (with general arrival statistics) can be solved. However, this needs the introduction of an heavy mathematical formalism, and for sake of clearness this generalization will not be consider here.

This paper is organized as follows. In the next section the priority assignment model is adapted to the Markov decision setting and the main result is stated. In section III, a related unconstrained problem is posed and, with the aid of dynamic programming, its optimal policies are characterized. In section IV, the unconstrained optimal policies are investigated with a special emphasis on the strict-priority rules. Using analytic function theory, the steady-state generating function for the queue lengths and mean line lengths are derived. These results enable us to exhibit the specific form of the constrained optimal policy in section V. In section VI, the optimal bias factor is obtained when the arrival process is globally geometric by resolving a Dirichlet boundary value problem.

II - MARKOV DECISION MODEL AND STATEMENT OF MAIN RESULT

In this section, the queueing system described in the introduction is adapted to a Markov decision framework. In order to do this, we must specify the state space, the decision space and the probabilistic evolution of the state and decision processes [9], [17], [8].

The state process $\{N_t, t = 1, 2, \ldots\}$ is defined on the state space I^2, where I denotes the non-negative integers. Employing the notation $N_t=(N_t(0), N_t(1))$, then $N_t(i)$ represents the number of class i customers in the queueing system at the beginning of slot $[t, t+1)$. The decision process $\{D_t, t = 1,2,\ldots\}$ is defined on the decision space $\{0,1\}$. The customer class i is selected for service during the slot $[t, t+1)$ if $D_t=i$.

A rule for making priority decisions is called a policy. In order to define a policy precisely, first let

$$H_t = \{N_1, D_1, N_2, D_2, \ldots, N_{t-1}, D_{t-1}, N_t\}$$

be the <u>history</u> of the Markov decision process ; such a history consists not only of the line-length process up to the present, but also the priority decisions D_s at the respective time slots. Then a policy is a sequence $v = (v_1, v_2, \ldots)$ where v_t is a probability measure on the decision space depending on the history H_t. This definition of a policy is consistent with those found in the literature [8], [9]. Each probability measure on $\{0,1\}$ is uniquely determined by the mass it assigns to $\{1\}$, which is a real number in $[0,1]$. Therefore, we can (and henceforth will) regard v_t as a $[0,1]$-valued function of H_t.

When policy v is employed, denote P_v for the probability measure governing the statistical evolution of the state and decision processes. P_v is to be defined on the probability space $(\Omega, \mathbf{F})$ supporting the random sequences $\{N_t\}$, $\{D_t\}$. The equations

$$P_v(D_t=1 \mid H_t) = v_t(H_t) \tag{2.2a}$$

$$P_v(D_t=0 \mid H_t) = 1 - v_t(H_t) \tag{2.2b}$$

partially define the measure P_v ; from them it is clear that the statistical evolution of the decision process $\{D_t\}$ is determined by the policy. The value that v_t takes on has the interpretation of being the probability that service is offered to the class-1 customer for the time-slot $[t, t+1)$.

To complete the definition of P_v, me must specify the conditional probabilities $P_v(N_{t+1} = n \mid H_t, D_t)$ with n belonging to I^2. To this end, introduce an auxilary probability triple $(\Omega', \mathbf{F}', \mathbf{P})$ supporting the I-valued random variables A_0, A_1, and the $\{0,1\}$-valued random variables B_0, B_1. The measure P is required to posses the following properties : B_0 and B_1 are independent of A_0 and A_1 with $P(B_i=1) = \mu_i$; A_0 and A_1 have the joint distribution corresponding to that of the arrivals of the two customer classes within a given slot. It is also convenient to introduce the transition mappings T_0 and T_1 from I^2 to the space of I^2-valued random vectors on $(\Omega', \mathbf{F}', \mathbf{P})$ by

$$T_i(n) = (n_0+A_0-(1-i)B_0 1(n_0>0),\ n_1+A_1-iB_1 1(n_1>0)) \qquad (2.3)$$

where $n = (n_0, n_1)$ and $1(.)$ is the indicator function. $T_i(n)$ is the state of the system at time $t + 1$ when the state and action chosen at time t are n and i, respectively. We may now complete the definition of P_v by taking

$$P_v(N_{t+1}=n \mid N_t=m,\ D_t=i,\ H_t) = P(T_i(m) = n). \qquad (2.4)$$

Note that the right-hand side of equation (2.3) is independent of the policy v, the history H_t and the epoch t, as required in the definition of a Markov decision process [9], [17]. We have thus constructed a Markov decision process, with finite decision and countable state space, that models the priority queueing system described in the introduction.

We now turn to the optimization objectives. For any policy v, there corresponds the long-run average line-lengths

$$K_v(n) = \overline{\lim_{t\to\infty}}\ \frac{1}{t}\, E_v\left[\sum_{s=1}^{t} N_s(0) \mid N_1=n\right] \qquad (2.5a)$$

$$C_v(n) = \overline{\lim_{t\to\infty}}\ \frac{1}{t}\, E_v\left[\sum_{s=1}^{t} N_s(1) \mid N_1=n\right] \qquad (2.5b)$$

where E_v is the expectation operator for P_v. A policy v is said to be <u>feasible</u> if $K_v(n) \le \alpha$ for all n in I^2, with α a fixed real number. A feasible policy v is <u>optimal</u> if

$$C_v(n) \le C_u(n) \qquad (2.6)$$

for all feasible policies u and all n in I^2.

An optimal policy therefore minimizes the global cost $C_u(n)$ subject to the global constraint $K_u(n) \le \alpha$ for all n. If an optimal policy exists, it may not be unique. Moreover, it may be difficult to implement since policies are in general <u>past</u>- and <u>time-dependent</u>. We are therefore motivated to search for an optimal policy with a simple structure. Less complex policies are now introduced to facilitate this search.

An important class of policies are those termed <u>non-idling</u>. For such policies, the server is never idle when there are customers in the system. More precisely, for all $t \geq 1$

$$v_t(H_t) = \begin{cases} 1 & \text{on } \{N_t(0) = 0,\ N_t(1) > 0\} \\ 0 & \text{on } \{N_t(0) > 0,\ N_t(1) = 0\}. \end{cases} \qquad (2.7)$$

It is reasonable to expect an optimal policy to be non-idling.

Of special interest is a much more restricted class which is termed <u>Simply Randomized</u> (SR) policies. These are obtained as follows : when there is only one customer class in the system, the policy behaves in a non-idling fashion ; when both classes are present, service is offerred to a customer class according to the toss of a biased coin. An SR policy is defined mathematically by

$$V_t(H_t) = \begin{cases} q & \text{on } \{N_t(0) > 0,\ N_t(1) > 0\} \\ 1 & \text{on } \{N_t(0) = 0,\ N_t(1) > 0\} \\ 0 & \text{on } \{N_t(0) > 0,\ N_t(1) = 0\} \end{cases} \qquad (2.8)$$

for all $t \geq 0$ where q belongs to $[0,1]$. Clearly, every SR policy is uniquely represented by a real number q belonging to $[0,1]$. With a slight abuse of notation, we shall often write f_q (or simply q) for the SR policy with bias factor q.

An important property of an SR policy is that the line-length process $\{N_t\}$ that it governs becomes a homogeneous Markov chain. This fact is readily verified by combining (2.2) and (2.4) with (2.8).

Although f_0 and f_1 are termed SR policies, they are not randomized but instead correspond to <u>strict-priority rules</u>. At the beginning of each slot, policy f_1 alway offers service to a customer of class 1 when such a customer is present in the system ; otherwise, f_1 offers service to a customer of class 0.

In order to state the main proposition that shall be proved in the forthcoming sections, we need to introduce some notations. Denote E for the expectation operator corresponding to P, and for i,j = 0,1 denote

$$\lambda_i = E[A_i] \tag{2.9}$$

$$\sigma_{ij}^2 = E[(A_i - \lambda_i)(A_j - \lambda_j)] \tag{2.10}$$

$$\rho = \frac{\lambda_0}{\mu_0} + \frac{\lambda_1}{\mu_1}. \tag{2.11}$$

Throughout we require λ_i and μ_i to be finite and positive and σ_{ij} to be finite for i,j = 0,1.

Also denote

$$\beta_0 = \frac{\sigma_{00}^2 + \lambda_0(1-\lambda_0)}{2(\mu_0 - \lambda_0)} \tag{2.12}$$

$$\beta_1 = \frac{\frac{\lambda_0\mu_0}{\mu_1-\lambda_1}\left[\sigma_{11}^2+\lambda_1(1-\lambda_1)\right]+\mu_1\sigma_{00}^2+2\mu_0\sigma_{01}^2+\mu_1\lambda_0(1-\lambda_0)-2\lambda_0\lambda_1\mu_0}{2\mu_0\mu_1(1-\rho)}. \tag{2.13}$$

The main result is

Theorem 2.1 Suppose $\rho < 1$. If $\alpha < \beta_0$ then there does not exist a feasible policy. If $\beta_0 \leq \alpha \leq \beta_1$ then there exists an optimal SR policy. If $\beta_1 \leq \alpha$ then the strict priority rule f_1 is optimal. □

The proof of this theorem shall be given in section V. In sections III and IV we develop the machinary needed for the proof while simultaneously obtaining other interesting results.

III - AN UNCONSTRAINED PROBLEM

Classical Markov decision theory [9], [17], [8] cannot be applied to the problem at hand due to the global constraint. In this section we study a related unconstrained problem to which we can apply the classical theory and which leads to the proof of the theorem 2.1.

For this reason, let $\gamma = \frac{\mu_1}{\mu_0}$ and introduce the finite-horizon cost

$$J_v^t(n) = E_v\left[\sum_{s=1}^{t} \gamma N_s(0) + N_s(1) \;\middle|\; N_1 = n\right] \tag{3.1}$$

along with the minimal cost

$$J^t(n) = \inf_v J_v^t(n). \tag{3.2}$$

For t-horizon problems, a policy $v = (v_1, v_2, \ldots, v_{t-1})$ is now a finite sequence, but otherwise defined in exactly the same manner as in the infinite-horizon case.

Observe that equations (3.1)-(3.2) define an optimization criterion for a finite-horizon Dynamic Programming (DP) problem with finite decision and countable state space. Moreover, the conditions for the semicontinuous model [9,p.51] are satisfied. We may therefore evoke the well-known results of DP. In particular, the DP equation [9,eq.2.5.14]

$$\begin{aligned} J^{s+1}(n) &= n_1+\gamma n_0+\min\{EJ^s(T_0n),\ EJ^s(T_1n)\} \\ &= n_1+\gamma n_0+EJ^s(T_0n) + \min_{i\in\{0,1\}} iE[J^s(T_1n) - J^s(T_0n)] \end{aligned} \tag{3.3}$$

is valid with $1 \leq s \leq t-1$. The second equality in (3.3) follows immediately from the first and the linearity property of the expectation operator. Furthermore, if a policy $v = (v_1, \ldots, v_{t-1})$ satisfies

$$v_{t-s}(H_{t-s})E[J^s(T_1n)-J^s(T_0n)] = \min_{i\in\{0,1\}} iE[J^s(T_1n) - J^s(T_0n)] \tag{3.4}$$

on $\{N_{t-s} = n\}$ for $1 \le s \le t-1$ and for all n in I^2, then v is optimal for the t-horizon problem. We remark that the optimality principle given by (3.4) is slightly different from what is given by most text-books because the policy v may be past-dependent and/or randomized. However, a glance at the derivation of (3.4) (e.g. [9 , section 1.6]) shows that the optimality principle extends to such policies.

Theorem 3.1 For all $t \ge 1$ the following holds

$$J^t(n_0,n_1) < J^t(n_0,n_1+1) \qquad n_0 \ge 0,\ n_1 \ge 0 \tag{3.5}$$

$$J^t(n_0,n_1) < J^t(n_0+1,n_1) \qquad n_0 \ge 0,\ n_1 \ge 0 \tag{3.6}$$

$$\mu_1[J^t(n_0,n_1)-J^t(n_0,n_1-1)]=\mu_0[J^t(n_0,n_1)-J^t(n_0-1,n_1)] \quad n_0\ge 1,\ n_1\ge 1. \tag{3.7}$$

Proof It is easily seen that the theorem is valid for $t = 1$. Suppose (3.5)-(3.7) hold with $t = s$. We must show (3.5)-(3.7) remain to hold for $t = s + 1$. In view of the inductive hypothesis (3.5), it easily follows that

$$E[J^s(T_1n) - J^s(T_0n)] \le 0 \tag{3.8}$$

when $n_0 = 0$. Similarly, form (3.6) we obtain

$$E[J^s(T_0n) - J^s(T_1n)] \le 0 \tag{3.9}$$

when $n_1 = 0$. If both $n_0 > 0$ and $n_1 > 0$ then

$$\begin{aligned} EJ^s(T_1n) &= \mu_1 EJ^s(n_0+A_0,n_1+A_1-1)+(1-\mu_1)EJ^s(n_0+A_0,n_1+A_1) \\ &= EJ^s(n_0+A_0,n_1+A_1)+\mu_1 E[J^s(n_0+A_0,n_1+A_1-1)-J^s(n_0+A_0,n_1+A_1)] \\ &= EJ^s(n_0+A_0,n_1+A_1)+\mu_0 E[J^s(n_0+A_0-1,n_1+A_1)-J^s(n_0+A_0,n_1+A_1)] \\ &= EJ^s(T_0n) \end{aligned} \tag{3.10}$$

where the inductive hypothesis (3.7) was used to obtain the third equality. Combining (3.8)-(3.10) gives

$$\min_{i\in\{0,1\}} iE[J^s(T_1 n)-J^s(T_0 n)] = \begin{cases} E[J^s(T_1 n)-J^s(T_0 n)] & n_1>0 \\ 0 & n_1 = 0. \end{cases} \tag{3.11}$$

Direct substitution of (3.10) and (3.11) into (3.3) yields the important relation

$$J^{s+1}(n_0,n_1) = \begin{cases} n_1+\gamma n_0 + EJ^s(n_0+A_0,n_1+A_1-B_1) & n_1 > 0 \\ n_1+\gamma n_0 + EJ^s(n_0+A_0-B_0,n_1+A_1) & n_0 > 0 \\ EJ^s(A_0,A_1) & n_0 = n_1 = 0. \end{cases} \tag{3.12}$$

Now from (3.12) we readily obtain

$$J^{s+1}(n_0,n_1+1) - J^{s+1}(n_0,n_1) =$$

$$\begin{cases} 1+E[J^s(n_0+A_0,n_1+1+A_1-B_1)-J^s(n_0+A_0,n_1+A_1-B_1)] & n_1 > 0 \\ 1+E[J^s(n_0+A_0-B_0,A_1+1)-J^s(n_0+A_0-B_0,A_1)] & n_1 = 0,\ n_0 > 0 \\ 1+E[J^s(A_0,1+A_1-B_1)-J^s(A_0,A_1)] & n_0 = n_1 = 0 \end{cases} \tag{3.13}$$

which, in light of the inductive hypothesis (3.5), is never less than 1 for all n in I^2. Thus (3.5) is established with $t = s+1$. In an analogous manner, the validation of (3.6) with $t = s+1$ can be easily carried out.

It remains to establish condition (3.7) for $t = s+1$. Fix n with $n_0 > 0$ and $n_1 > 0$. On one hand equation (3.12) implies

$$J^{s+1}(n_0,n_1) - J^{s+1}(n_0,n_1-1)$$

$$= 1 + E[J^s(n_0+A_0-B_0,n_1+A_1) - J^s(n_0+A_0-B_0,n_1+A_1-1)]$$

$$= 1 + \mu_0 E[J^s(n_0+A_0-1,n_1+A_1)-J^s(n_0+A_0-1,n_1+A_1-1)]$$

$$+ (1-\mu_0)\ E[J^s(n_0+A_0,n_1+A_1) - J^s(n_0+A_0,n_1+A_1-1)]. \qquad (3.14)$$

On the other hand, (3.12) also implies

$$J^{s+1}(n_0,n_1) - J^{s+1}(n_0-1,n_1)$$

$$= \frac{\mu_1}{\mu_0} + \mu_1\ E[J^s(n_0+A_0,n_1+A_1-1) - J^s(n_0+A_0-1,n_1+A_1-1)]$$

$$+ (1-\mu_1)\ E[J^s(n_0+A_0,n_1+A_1) - J^s(n_0+A_0-1,n_1+A_1)]. \qquad (3.15)$$

Multiplying both sides of (3.14) by μ_1 and both sides of (3.15) by μ_0, and using the inductive hypothesis (3.7), condition (3.7) follows for $s = t+1$.

$\square$

In order to state the main result of this section, introduce the long-run average unconstrained cost

$$J_v(n) = \overline{\lim_t}\ \frac{1}{t} J_v^t(n) \qquad (3.16)$$

and corresponding minimal cost

$$J(n) = \inf_v J_v(n). \qquad (3.17)$$

The policy v is said to be <u>optimal for the unconstrained problem</u> if

$$J_v(n) = J(n) \tag{3.18}$$

for all $n \in I^2$.

Theorem 3.2 Every non-idling policy is optimal for the unconstrained problem. □

Proof From (3.10) and the definition (2.7) of a non-idling policy v we have

$$V_{t-s}(H_{t-s})E[J^s(T_1 n)-J^s(T_0 n)] = \begin{cases} E[J^s(T_1 n)-J^s(T_0 n)] & \text{on } \{N_{t-s}(1) > 0\} \\ 0 & \text{on } \{N_{t-s}(1) = 0\}. \end{cases} \tag{3.19}$$

Combining (3.19) and (3.11) with the optimality principle (3.4) we get for all $n \in I^2$

$$J_v^t(n) \leq J_u^t(n) \tag{3.20}$$

for any policy u. Since (3.20) holds for all $t \geq 0$, it follows that

$$\overline{\lim}\ \frac{1}{t} J_v^t(n) \leq \overline{\lim}\ \frac{1}{t} J_u^t(n) \tag{3.21}$$

from which the theorem immediately follows. □

This completes our discussion of the unconstrained problem. It will turn out to be crucial for the proof of theorem 2.1. Before doing this, however, we examine in the next section the strict priority rules.

IV - ANALYTIC ANALYSIS FOR THE STRICT PRIORITY RULES

We begin this section by studying the equilibrium behavior of the two-dimensional generating function

$$F_q(x,y;t) \triangleq E_q[x^{N_t(1)}\, y^{N_t(0)} \mid N_1 = m] \tag{4.1}$$

corresponding to the SR policy f_q and fixed initial state $m \in I^2$. The generating functions

$$G(x,y) \triangleq \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} P(A_0 = j,\ A_1 = k)\, x^k y^j \tag{4.2}$$

and $F_q(x,y;t)$ are regular in $|x| < 1$, $|y| < 1$ and continuous in $|x| \leq 1$, $|y| \leq 1$. We may now state the fundamental

Theorem 4.1 Suppose that for fixed q

$$\lim_{t\to\infty} F_q(x,y;t) = F_q(x,y) \tag{4.3}$$

exists for all $|x| \leq 1$, $|y| \leq 1$. Then $F_q(x,y)$ satisfies the functional equation

$$T_q(x,y)F_q(x,y) = G(x,y)[R(x,y)\{qF_q(0,y)-(1-q)F_q(x,0)\}+S_q(x,y)F_q(0,0)] \tag{4.4}$$

where

$$T_q(x,y) \triangleq 1-G(x,y)[1-q\mu_1(1-\frac{1}{x}) - (1-q)\,\mu_0(1-\frac{1}{y})] \tag{4.5}$$

$$S_q(x,y) \triangleq (1-q)\,\mu_1(1-\frac{1}{x}) + q\mu_0(1-\frac{1}{y}) \tag{4.6}$$

$$R(x,y) \triangleq \mu_1(1-\frac{1}{x}) - \mu_0(1-\frac{1}{y}). \tag{4.7}$$

Proof For any policy v, by the definitions (2.2) and (2.4) the following holds on $\{N_t=m\}$

$$P_v(N_{t+1} = n \mid H_t) = v_t(H_t)P(T_1(m)=n)+(1-v_t(H_t))P(T_0(m)=n) \qquad (4.8)$$

This implies, along with the definition (2.3) of $T_i(m)$, that on $\{N_t=m\}$

$$E_v[x^{N_{t+1}(1)}\, y^{N_{t+1}(0)} \mid H_t] = v_t(H_t)E[x^{m_1+A_1-B_1 1(m_1>0)}\, y^{m_0+A_0}]$$

$$+ (1-v_t(H_t))E[x^{m_1+A_1}\, y^{m_0+A_0-1(m_0>0)B_0}]$$

$$= G(x,y)\ \{v_t(H_t)E[x^{m_1-B_1 1(m_1>0)}\, y^{m_0}]$$

$$+ (1-v_t(H_t))E[y^{m_0-1(m_0>0)\,B_0}\, x^{m_1}]\}. \qquad (4.9)$$

The second equality in (4.9) is a consequence of the independence of A_0, A_1 from B_0, B_1. Specializing to the SR policy f_q and taking expectations of both sides in (4.9), a straightforward but tedious calculation yields

$$F_q(x,y;t) = G(x,y)\Big\{[1-q\mu_1(1-\frac{1}{x}) - (1-q)\mu_0(1-\frac{1}{y})]\, F_q(x,y;t)$$

$$+ R(x,y)[qF_q(0,y;t)-(1-q)F_q(x,0;t)] + S_q(x,y)F(0,0;t)\Big\}. \quad (4.10)$$

The desired result follows from (4.10) after taking limits on both sides and rearranging terms. $\square$

Recall that under any SR policy f_q, the state process $\{N_t\}$ is a homogeneous Markov chain ; denote Q_q for the corresponding transition matrix. For the remainder of this article we assume that Q_0 and Q_1 are irreducible. This holds, in particular, when A_0 and A_1 are independent along with $P(A_i = 0) > 0$, $P(A_i > 1) > 0$. This assumption is quite unnecessary and is used only to avoid superficial complexity. Note that this assumption implies, with $\theta = (0,0)$,

$$Q_i(\theta,\theta) = P(A_0 = 0,\ A_1 = 0) > 0 \quad i = 0,1 \tag{4.11}$$

so that Q_0 and Q_1 are also aperiodic.

For the remainder of this section we specialize to the strict priority rule f_0 ; the analytic analysis for the general SR policy will be taken up in section VII. For convenience, we write (in this section only) Q and F for Q_0 and F_0, respectively.

Theorem 4.2 Q is positive recurrent iff $\rho < 1$. When $\rho < 1$, $F(x,y)$ is well-defined by (4.3), independent of the initial condition m and given by

$$F(x,y) = \sum_{n_0=0}^{\infty} \sum_{n_1=0}^{\infty} \pi(n_0,n_1)\, x^{n_1} y^{n_0} \qquad |x| \le 1,\ |y| \le 1 \tag{4.12}$$

where π is the unique invariant probability measure on I^2 for Q. $\square$

Proof The first statement is a consequence of the irreducibility of Q and theorem 4.5 of [2] (this result can also be obtained from [1], [16]). A well-known result from Markov chain theory then gives for $m,n \in I^2$

$$\lim_{t\to\infty} Q^{(t)}(m,n) = \pi(n_0,n_1) \tag{4.13}$$

when $\rho < 1$. The remaining statements follow from (4.13) since

$$F(x,y;t) = \sum_{n_0=0}^{\infty} \sum_{n_1=0}^{\infty} x^{n_1} y^{n_0}\, Q^{(t)}(m,n). \square \tag{4.14}$$

Observe (4.12) implies that $F(x,y)$ is regular in the polydisk $|x| < 1$, $|y| < 1$ and continuous in $|x| \le 1$, $|y| \le 1$. Multiplying both sides of (4.4) by y and setting $q = 0$, we obtain

$$T(x,y)F(x,y)=G(x,y)\left\{\left[\mu_0(y-1)-\mu_1 y\left(1-\frac{1}{x}\right)\right]F(x,0)+\mu_1 y\left(1-\frac{1}{x}\right)F(0,0)\right\} \tag{4.15}$$

where

$$T(x,y) \triangleq y - G(x,y)(y-\mu_0 y+\mu_0). \tag{4.16}$$

Lemma 4.2 Under the ergodicity condition $\rho < 1$, for every x with $|x| < 1$ the equation $T(x,y) = 0$ has a unique root $y = y(x)$ in the unit circle $|y| < 1$. Moreover, $y(x)$ defines a regular function in $|x| < 1$. □

Proof For $|x| < 1$, $|y| = 1$ we have

$$|G(x,y)(y-\mu_0 y+\mu_0)| \le |G(x,y)| < 1 = |y|. \tag{4.17}$$

From Rouché's theorem, we conclude that for $|x| < 1$ the equation $T(x,y) = 0$ has a unique root $y = y(x)$ in the unit circle $|y| < 1$. The fact that $y(x)$ is regular in $|x| < 1$ follows from the implicit function theorem for complex variables, which applies here since $\frac{\partial}{\partial y}T(x,y) \ne 0$ for $|x| < 1$ [10, Th.7.1, p.101]. □

In the following theorem, we obtain the generating function $F(x,y)$ in terms of $G(x,y)$, μ_0 and μ_1 ; in principle, the invariant probability measure $\pi(.,.)$ can be obtained by inverting $F(x,y)$. For the theorem, and what follows, for z on the unit circle $|z| = 1$ introduce the notation

$$y^{(k)}(z) = \lim_{x \to z} y^{(k)}(x) \tag{4.18}$$

if the limit exists from within the circle $|x| < 1$. Here $y^{(k)}(x)$ is the k^{th} derivative of $y(x)$.

Theorem 4.3 For $\rho < 1$ the following holds on $|x| \leq 1$, $|y| \leq 1$

$$F(x,y) = \frac{F(0,0)G(x,y)\mu_0\mu_1(1 - \frac{1}{x})(\frac{y}{y(x)} - 1)}{T(x,y)\ R(x,y(x))} . \tag{4.19}$$

Moreover, $F(0,0) = 1 - \rho$ and $y(1) = 1$. $\square$

Proof Let D be the set in the complex plane defined by

$$D = \{x : |x| < 1,\ y(x) \neq 0\}. \tag{4.20}$$

First note that $G(x,y(x)) \neq 0$ for $x \in D$. Because of the analycity of $F(x,y)$ in $|x| < 1$, $|y| < 1$, it follows that the right-hand side of (4.15) vanishes when the kernel $T(x,y)$ vanishes. For $x \in D$, this gives

$$F(x,0) = \frac{\mu_1(1 - \frac{1}{x})\ F(0,0)}{\mu_1(1 - \frac{1}{x}) - \mu_0(1 - \frac{1}{y(x)})} . \tag{4.21}$$

Because $y(x)$ is regular in $|x| < 1$ and because $y(0) \neq 0$ (since $G(0,0) > 0$), it follows from a well-known result of analytic function theory that $y(x) = 0$ only at isolated points in $|x| < 1$. This implies, along with the continuity of $F(x,0)$ in $|x| \leq 1$ and the continuity of $y(x)$ in $|x| < 1$ that (4.21) holds everywhere in $|x| \leq 1$. Putting (4.21) into (4.15), we get (4.19). It is shown in appendix I that $F(0,0) = 1 - \rho$. It remains to prove $y(1) = 1$. To this end, observe that $\pi(.,.)$ is strictly positive and so is

$$\lim_{x \to 1} F(x,0) = \sum_{k=0}^{\infty} \pi(0,k). \tag{4.22}$$

This gives the desired result after taking the corresponding limit on both sides of (4.21). $\square$

Recall the definitions (2.12) and (2.13) of β_0 and β_1. Let β be given by the right-hand side of (2.13) with the indicies 0 and 1 interchanged.

Corollary 4.4 If $\rho < 1$, the mean line-lengths are given by

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} j\pi(j,k) = \beta_0 \tag{4.23}$$

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} k\pi(j,k) = \beta. \quad \square \tag{4.24}$$

Proof It is sufficient to show

$$\frac{\partial}{\partial y} F(x,y) \Big|_{x=1,y=1} = \beta_0 \tag{4.25}$$

$$\frac{\partial}{\partial x} F(x,y) \Big|_{x=1,y=1} = \beta. \tag{4.26}$$

To this end, we first obtain

$$y^{(1)}(1) = \frac{\lambda_1}{\mu_0 - \lambda_0} \tag{4.27}$$

$$y^{(2)}(1) = \frac{\lambda_0(1+\lambda_0-2\mu_0+\sigma_{00}^2)y^{(1)}(1)^2+2[\sigma_{01}^2+\lambda_1(1-\mu_0+\lambda_0)]y^{(1)}(1)+\sigma_{11}^2+\lambda_1^2-\lambda_1}{(\mu_0 - \lambda_0)} \tag{4.28}$$

These two equations are established by differentiating

$$y(x) = G(x,y(x))\ (y(x) - \mu_0 y(x) + \mu_0) \tag{4.29}$$

in $|x| < 1$ and using $y(1) = 1$. Taking into account (4.27)-(4.28), a straightforward but tedious differentiation of (4.19) then yields (4.25)-(4.26). $\square$

We remark that the case corresponding to the strict priority rule f_1 is completely analogous to that of f_0. In fact, it is easily seen that for $\rho < 1$

$$\frac{\partial}{\partial y} F_1(x,y)\Big|_{x=1,y=1} = \beta_1. \tag{4.30}$$

V - OPTIMALITY OF SR POLICIES

In this section we combine the results established in sections III and IV to prove theorem 2.1. We will also discuss the non-ergodic case $\rho \geq 1$.

Recall that Q_q is the transition matrix for the Markov chain $\{N_t\}$ when driven by the SR policy f_q. Because Q_0 and Q_1 are assumed to be irreducible and because

$$Q_q = qQ_1 + (1-q)Q_0 \tag{5.1}$$

it follows that Q_q is irreducible. When Q_q is positive recurrent, we write π_q for the unique probability measure π on I^2 that satisfies $\pi = \pi Q_q$.

Let g be any non-negative function on I^2. It is a well-known result from the theory of Markov chains [13, page 51] that

$$\lim_t \frac{1}{t} E_q\left[\sum_{s=0}^{t} g(N_s) \,\Big|\, N_1 = n\right] = \sum_i \sum_j g(i,j)\, \pi_q(i,j) \tag{5.2}$$

if Q_q is positive recurrent. When $\rho < 1$, we may use (5.2) and corollary 4.4 to express

$$J_{f_0}(n) = \sum_{i,j} \pi_0(i,j)\,(i\gamma+j) = \gamma\beta_0 + \beta \overset{\Delta}{=} M \tag{5.3}$$

which is finite. Furthermore, by theorem 3.2, equation (5.3) implies

$$J_v(n) = M \tag{5.4}$$

for all n in I^2 and all non-idling policies v.

Lemma 5.1 Q_q is positive recurrent for each q in [0,1] iff $\rho < 1$. $\square$

Proof Appendix I shows that $\rho < 1$ when Q_q is positive recurrent. Now, fix $q \in [0,1]$ and let $L > M$. If we denote $Z_s = N_s(1) + \gamma N_s(0)$, then

$$\frac{1}{t}\sum_{s=1}^{t} Z_s \geq \frac{1}{t}\sum_{s=1}^{t} L\, 1\,(Z_s \geq L)$$

$$= L - \frac{L}{t}\sum_{s=1}^{t} 1(Z_s < L)$$

$$= L - \sum_{i\gamma+j<L} \frac{L}{t}\sum_{s=1}^{t} 1(N_s(0)=i,\ N_s(1) = j). \tag{5.5}$$

When Q_q is not positive recurrent, we have [13, page 25]

$$\lim_{t} \sum_{s=1}^{t} \frac{1}{t} P_q(N_s(0)=i,\ N_s(1)=j) \mid N_1=n) = 0 \tag{5.6}$$

which in turn implies

$$\overline{\lim_{t}}\ \frac{1}{t} E_q[\sum_{s=1}^{t} Z_s \mid N_1 = n] \geq L > M. \tag{5.7}$$

But this contradicts (5.4). $\square$

An important consequence of the previous lemma and equations (5.2), (5.4) is

$$\sum_{i,j} (\gamma i+j)\ \pi_q(i,j) = M \tag{5.8}$$

for all SR policies f_q whenever $\rho < 1$. The following lemma and theorem address the continuity of $q \to \pi_q$ and $q \to K_q$, respectively. These results will lead to the proof of theorem 2.1.

Lemma 5.2 $q \to \pi_q(n)$ is continuous on $[0,1]$ for each $n \in I^2$. □

Proof It is not difficult to show using (5.8) that the family of probability measures $\{\pi_q : q \in [0,1]\}$ is tight [6]. Let $\{q_k\}$ be a sequence in $[0,1]$ converging to q_0 and write Q_k and π_k for Q_{q_k} and π_{q_k}, respectively. By the tightness of $\{\pi_q : q \in [0,1]\}$ there is a subsequence of $\{\pi_k\}$, that we write again as $\{\pi_k\}$, and a measure π on I^2 such that

$$\lim_k \pi_k(n) = \pi(n) \tag{5.9}$$

for all $n \in I^2$ [6, theorem 6.1]. It suffices to show that $\pi = \pi_0$ or equivalently that $\pi = Q_0\pi$. But

$$\pi - \pi Q_0 = (\pi - \pi_k) + (\pi_k - \pi)Q_k + \pi(Q_k - Q_0). \tag{5.10}$$

The three terms on the right-hand side of (5.10) become arbitrarily small pointwise as $k \to \infty$. Indeed, the first term tends to zero pointwise by (5.9) and the second term by Scheffé's theorem [6, p.224]. For the third term of (5.10) we simply note that $q \to Q_q(m,n)$ is continuous (see equation (5.1)) and then employ Lebesque's dominated convergence theorem.

□

Theorem 5.3 If $\rho < 1$ then $q \to K_q(n)$ is continuous on [0,1]. $\square$

Proof For all $n \in I^2$, equation (5.2) implies

$$K_q(n) = \sum_j \sum_k j\pi_q(j,k). \tag{5.11}$$

Therefore, by the previous lemma and [6, equation (5.2) and theorem 5.4] it sufficies to show that

$$\sum_{j\geq\ell} \sum_{k\geq 0} \pi_q(j,k) \leq \sum_{j\geq\ell} \sum_{k\geq 0} \pi_1(j,k) \tag{5.12}$$

for all $q \in [0,1]$ and all $\ell \geq 0$. But (5.12) is equivalent to

$$\lim_t \frac{1}{t} \sum_{s=1}^{t} P_q(N_s(0) \geq \ell) \leq \lim_t \frac{1}{t} \sum_{s=1}^{t} P_1(N_s(0) \geq \ell). \tag{5.13}$$

Now, equation (5.13) will follow if we can prove for any $n \in I^2$ and for all $t \geq 1$ that f_1 maximizes

$$E_v[\sum_{s=1}^{t} 1(N_s(0) \geq \ell) \mid N_1 = n] \tag{5.14}$$

over the class of non-idling policies. However, this is obviously true and can be rigorously (and easily) proven by dynamic programming with state-dependent decision spaces. $\square$

Proof of theorem 2.1 By equation (5.11), corollary 4.4 and the remark at the end of section 4, we have

$$K_{f_i}(n) = \beta_i \qquad i = 0,1 \tag{5.15}$$

for all $n \in I^2$.

Suppose $\beta_0 \leq \alpha \leq \beta_1$. It then follows from theorem 5.3 and equation (5.15) that there is a $q \in [0,1]$ such that

$$K_{f_q}(n) = \alpha$$

for all $n \in I^2$. The SR policy f_q corresponding to this q is by definition feasible. Moreover, by theorem 3.2, we have

$$C_{f_q}(n) + \gamma\alpha = J_{f_q}(n)$$

$$\leq \overline{\lim_t} \frac{1}{t} E_v[\sum_{s=1}^{t} \gamma N_s(0) + N_s(1) \mid N_1 = n]$$

$$\leq C_v(n) + \gamma K_v(n) \tag{5.16}$$

for all $n \in I^2$ and all policies v. Consequently,

$$C_{f_q}(n) \leq C_v(n) \tag{5.17}$$

for all $n \in I^2$ and all feasible policies v. Hence, the SR policy f_q is optimal for the constrained problem.

For the other cases, first note that the strict priority rule f_i minimizes for all $n \in I^2$ and all $t \geq 1$

$$E_v[\sum_{s=1}^{t} N_s(i) \mid N = n] \tag{5.18}$$

over all policies v. This is easily established using dynamic programming. Therefore

$$K_{f_0}(n) \leq K_v(n) \tag{5.19}$$

$$C_{f_1}(n) \leq C_v(n). \tag{5.20}$$

Now, if $\alpha < \beta_0$, then from (5.15) and (5.19) we have

$$\alpha < K_v(n) \tag{5.21}$$

for all $n \in I^2$ and all policies v, i.e., all policies are infeasible. On the other hand, if $\beta_1 \leq \alpha$, then f_1 is feasible by (5.15) and optimal by (5.20). $\square$

<u>Remark</u> It is clear that $q \to K_q$ is strictly increasing and therefore there is a unique q such that $k_q = \alpha$ when $\beta_0 \leq \alpha \leq \beta_1$.

<u>Corollary 5.4</u> Suppose $\rho \geq 1$ and Q_0 and Q_1 are irreducible. If $\lambda_0 < \mu_0$ and $\beta_0 \leq \alpha$, then f_0 is optimal for the constrained problem. Otherwise, there does not exist a feasible policy. $\square$

<u>Proof</u> Using techniques in the proof of lemma 5.1, it can be shown that $J_{f_0}(n) = \infty$ for all $n \in I^2$. Theorem 3.2 then implies

$$J_v(n) = \infty \tag{5.22}$$

for all $n \in I^2$ and all policies v. Therefore, for any $n \in I^2$ and any policy v

$$K_v(n) = \infty \text{ and/or } C_v(n) = \infty. \tag{5.23}$$

It follows from (5.23) that any feasible policy is optimal for the constrained problem. But, by (5.19), there exists a feasible policy iff

$$K_{f_0}(n) \leq \alpha \tag{5.24}$$

for all $n \in I^2$.

The corollary will follow if we can show that $\lambda_0 < \mu_0$ and $\beta_0 \leq \alpha$ are necessary and sufficient conditions for (5.24). This is done by observing that under f_0, $\{N_t(0),\ t \geq 1\}$ is a Markov chain corresponding to a discrete-time queueing system with one customer class. Classical queueing analysis then shows that this Markov chain is positive recurrent iff $\lambda_0 < \mu_0$, and that $K_{f_0}(n) = \beta_0$ when $\lambda_0 < \mu_0$. This establishes the corollary. $\square$

VI DETERMINATION OF THE OPTIMAL BIAS FACTOR WHEN THE ARRIVAL PROCESS IS GLOBALLY GEOMETRIC

Consider the following discrete-time queueing system. Within slots customers arrive to the system from a single source according to a geometric distribution with parameter $\frac{\lambda}{\lambda+1}$, where $\lambda = \lambda_0 + \lambda_1$. There are two buffers in the system, and an arriving customer is routed to buffer i (i=0,1) with probability $\frac{\lambda_i}{\lambda}$. The routing mechanism is assumed to be independent of the arrival process (see figure 1). Consequently, the joint probability for the number of arrivals to buffers 0 and 1 within a given slot is

$$P(A_0=i,\ A_1=j) = \frac{1}{\lambda+1}\left(\frac{\lambda}{\lambda+1}\right)^{i+j}\binom{i+j}{i}\left(\frac{\lambda_0}{\lambda}\right)^{i}\left(\frac{\lambda_1}{\lambda}\right)^{j}. \quad (6.1)$$

We assume that the service requirements are independent from customer to customer and are geometrically distributed with buffer dependent parameter μ_i. It is clear that the queueing system just described is mathematically equivalent to the one described in the introduction with arrival distribution characterized by (6.1).

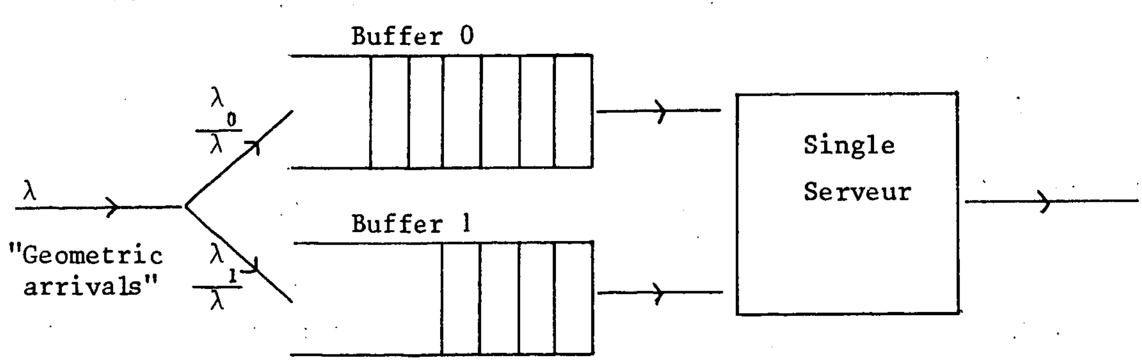

Figure 1

A standard calculation gives

$$G(x,y) = \frac{1}{1+\lambda_1(1-x)+\lambda_0(1-y)} . \tag{6.2}$$

Dividing equation (4.4) by G(x,y), and then employing (6.2), it is readily seen that (4.4) can be rewritten as

$$[\lambda_1(1-x)+\lambda_0(1-y)+q\mu_1(1-\frac{1}{x}) + (1-q)\mu_0(1-\frac{1}{y})]\, F_q(x,y)$$

$$= R(x,y)[qF_q(0,y)-(1-q)F_q(x,0)] + S_q(x,y)\, F_q(0,0). \tag{6.3}$$

It turns out that this functional equation is exactly the same as the one studied in [12, eq.1.2] up to an obvious change of notation. We may therefore evoke the following result [12, section 6].

Theorem 6.1 Under the ergodicity condition $\rho<1$, the solution of equation (6.3) is given for $q\theta(0,1)$, $|x| < \sqrt{\mu_1\lambda_1^{-1}q}$, $|y| < \sqrt{\mu_0\lambda_0^{-1}(1-q)}$ by

$$F_q(x,0) = \frac{2\sqrt{\lambda_1 q\mu_1}}{\pi} \int_0^{\pi} \frac{x\sin\theta\ \tau_q(\theta)\, d\theta}{q\mu_1+\lambda_1 x^2-2x\sqrt{\lambda_1 q\mu_1}\cos\theta} + 1-\rho \tag{6.4}$$

$$F_q(0,y) = \frac{2\sqrt{\lambda_0(1-q)\mu_0}}{\pi} \int_0^{\pi} \frac{y\sin\theta\ \nu_q(\theta)d\theta}{(1-q)\mu_0+\lambda_0 y^2-2y\sqrt{\lambda_0(1-q)\mu_0}\cos\theta} +1-\rho \tag{6.5}$$

where

$$\nu_q(\theta) = \frac{-\mu_1(1-\frac{1}{H(\theta)})\sqrt{\mu_0\lambda_0^{-1}(1-q)}\,(1-\rho)\sin\theta}{([\mu_0-\sqrt{\mu_0\lambda_0^{-1}(1-q)}\cos\theta - \mu_1(1-\frac{1}{H(\theta)})]^2+\frac{\lambda_0\mu_0}{1-q}\sin^2\theta)q} \tag{6.6}$$

$$H_q(\theta) = \frac{\lambda_1+\mu_1 q+I_q(\theta) - \sqrt{[(\sqrt{\lambda_1}+\sqrt{q\mu_1})^2+I_q(\theta)][(\sqrt{\lambda_1}-\sqrt{q\mu_1})^2+I_q(\theta)]}}{2\lambda_1}$$

$$I_q(\theta) = \lambda_0 + \mu_0(1-q) - 2\sqrt{\lambda_0(1-q)\mu_0}\ \cos\theta$$

and $\tau_q(\theta)$ is obtained from (6.6) by interchanging indicies 0 and 1 and replacing q by $1 - q$. □

The mean line-length K_{f_q} is now obtained as follows : differentiating (6.3) twice in the variable y, we get for $q \in (0,1)$ and $\lambda_0 \neq \mu_0(1-q)$

$$K_{f_q} = \frac{\partial}{\partial y} F(x,y)\Big|_{x=1,y=1} = \frac{q\,\frac{\partial}{\partial y} F_q(0,y)\big|_{y=1} - \frac{\lambda_0}{\mu_0}}{\frac{\lambda_0}{\mu_0} - (1-q)}. \qquad (6.7)$$

Similarly, differentiating (6.3) three times in the variable y yields for $q \in (0,1)$ and $\lambda_0 = \mu_0(1-q)$

$$K_{f_q} = \frac{1-q + q\left[\frac{1}{2}\,\frac{\partial^2}{\partial y^2} F_q(0,y)\big|_{y=1} - \frac{\partial}{\partial y} F_q(0,y)\big|_{y=1}\right]}{1-q}. \qquad (6.8)$$

For $\lambda_0 < \mu_0(1-q)$, the formula (6.5) together with (6.7) allows the determination of K_{f_q}.

Consider now the case $\lambda_0 \geq \mu_0(1-q)$, which implies $\lambda_1 < \mu_1 q$ under the ergodicity condition $\rho < 1$.

It is seen that in this case the formula (6.5) must be analytically continued to $\sqrt{\mu_0\lambda_0^{-1}(1-q)} \le |y| \le 1$ in order to derive $\frac{\partial}{\partial y} F_q(0,y)$ and $\frac{\partial^2}{\partial y^2} F_q(0,y)$ point at $y = 1$.

Using the lemma 2.2 in [12, p329] as well as the relation 4.4 [12, p334] (up to the change of notation), the analytic continuation is given by :

$$qF_q(0,y) = (1-q)\ F_q(K_q(y),0) - \frac{S_q(K_q(y),y)}{R_q(K_q(y),y)} F_q(0,0) \qquad (6.9)$$

for $\sqrt{\mu_0\lambda_0^{-1}(1-q)} \le |y| \le 1$ with

$$K_q(y) \overset{\text{def}}{=} \frac{\lambda_1+q\mu_1+A_q(y) - \sqrt{(\lambda_1+q\mu_1+A_q(y))^2-4\lambda_1 q\mu_1}}{2\lambda_1} \qquad (6.10)$$

$$A_q(y) \overset{\text{def}}{=} \lambda_0(1-y) + \mu_0(1-q)\ (1 - \frac{1}{y}). \qquad (6.11)$$

$F_q(K_q(y),0)$ in (6.9) is provided by formula (6.4), since for

$\sqrt{\mu_0\lambda_0^{-1}(1-q)} \le |y| \le 1$ then $|K_q(y)| < \sqrt{\mu_1\lambda_1^{-1}q}$ (see [12, lemma 2.2, p329]).

From equations (6.4)-(6.11) it is possible to obtain numerically the function $q \to K_{f_q}$ for $q \in (0,1)$, and hence to obtain the optimal bias factor for $\alpha \ge \beta_0$. We have done this for the case $\lambda_1 = 0.54$, $\lambda_0 = 0.26$, $\mu_1 = \mu_0 = 1$ with

$\alpha = \frac{\beta_0+\beta_1}{2}$ (see figure 2).

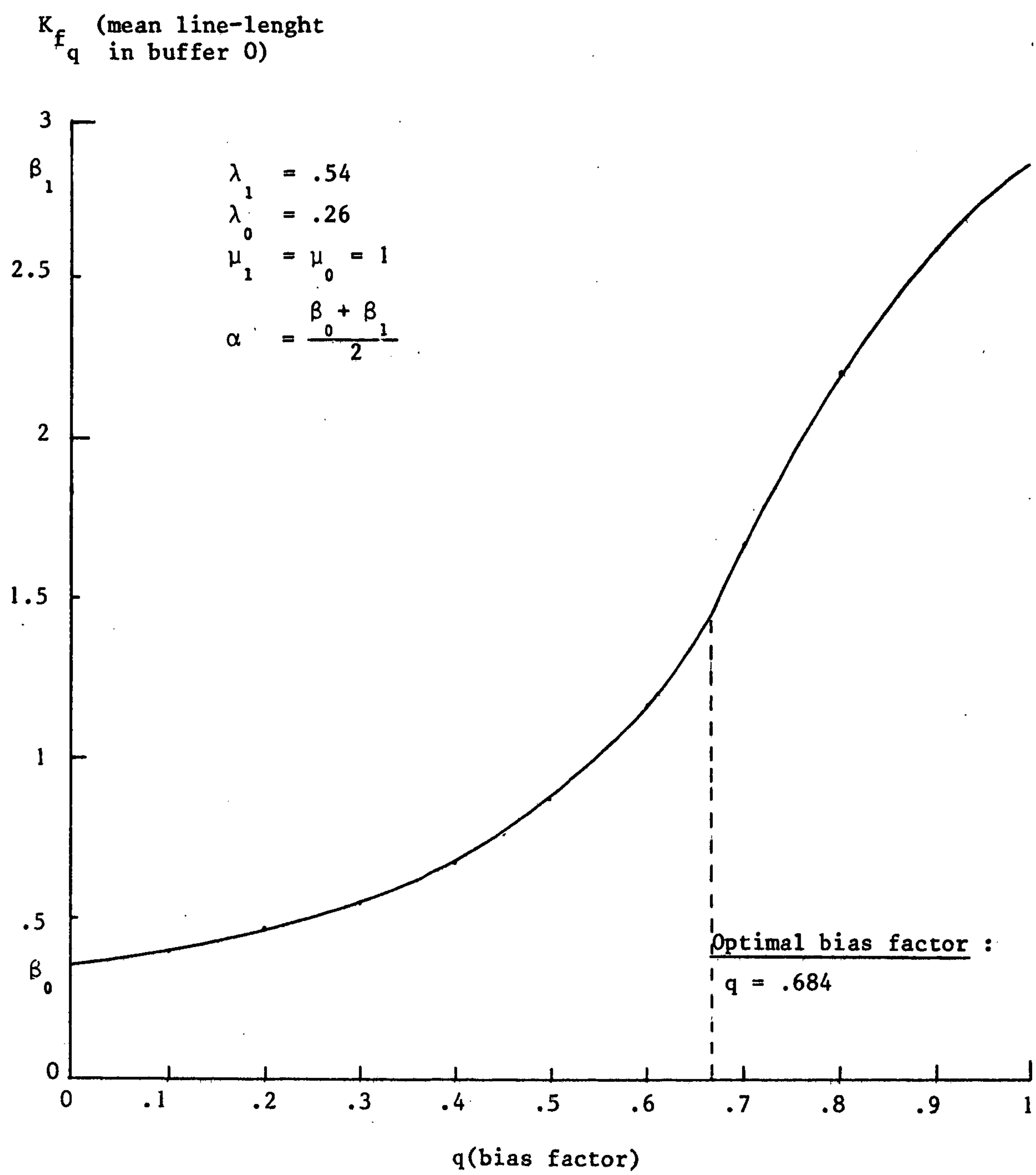

Figure 2: Optimal bias factor for a globally geometric arrival process.

APPENDIX I

Here we compute $F_q(0,0)$. Differentiating (4.4) in x for y = 1, then letting x = 1 yields

$$F_q(0,0) = \frac{(1-q)F_q(1,0) - qF_q(0,1) - \frac{\lambda_1}{\mu_1} + q}{1-q} .$$

Differentiating now (4.4) in y for x = 1, then letting y = 1 yields

$$F_q(0,0) = \frac{qF_q(0,1) - (1-q)\ F_q(1,0) - \frac{\lambda_0}{\mu_0} + 1-q}{q} .$$

From the above two equations we deduce that $F_q(0,0) = 1-\rho$.

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